DYNAMIC PROBLEMS FOR A TRANSVERSELY ISOTROPIC ELASTIC CYLINDER

(DINAMICHESKIE ZADACHI DLIA TRANSVERSAL'NOIZOTROPNOGO UPRUGOGO TSILINDRA)

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R. Ia. SUNCHELEEV
(Tashkent)

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Problems on steady oscillations of elastic isotropic media received a thorough treatment in [1 to 5] and other works. Some dynamic problems for nonhomogeneous and anisotropic media, were investigated in [6 to 8]. Present paper is concerned with obtaining a class of general solutions of dynamic problems of the theory of elasticity for a transversely isotropic cylinder.

1. Starting with the dynamic system of Lamé equations for a transversely isotropic homogeneous elastic medium.

$$\frac{\partial^2 U_1}{\partial t^2} = C_{66}\left(\frac{\partial^2 U_1}{\partial x_1^2} + \frac{\partial^2 U_1}{\partial x_2^2}\right) + C_{55}\frac{\partial^2 U_1}{\partial x_3^2} + \frac{\partial}{\partial x_1}\left[\frac{C_{11}+C_{12}}{2}\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_1}\right) + (C_{13}+C_{55})\frac{\partial U_3}{\partial x_3}\right] \tag{1.1}$$

$$\frac{\partial^2 U_2}{\partial t^2} = C_{66}\left(\frac{\partial^2 U_2}{\partial x_1^2} + \frac{\partial^2 U_2}{\partial x_2^2}\right) + C_{55}\frac{\partial^2 U_2}{\partial x_3^2} + \frac{\partial}{\partial x_2}\left[\frac{C_{11}+C_{12}}{2}\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}\right) + (C_{13}+C_{55})\frac{\partial U_3}{\partial x_3}\right]$$

$$\frac{\partial^2 U_3}{\partial t^2} = C_{55}\left(\frac{\partial^2 U_3}{\partial x_1^2} + \frac{\partial^2 U_3}{\partial x_2^2}\right) + C_{33}\frac{\partial^2 U_3}{\partial x_3^2} + (C_{13}+C_{55})\frac{\partial}{\partial x_3}\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}\right)$$

$$(C_{11} - C_{12} = 2C_{66})$$

we shall introduce cylindrical coordinates and make the following substitution of the sought functions

$$U_1 = \frac{1}{2}(e^{i\varphi} W_1 + e^{-i\varphi} W_2)\, U_2 = -\frac{1}{2}(e^{i\varphi} W_1 - e^{-i\varphi} W_2)\, U_3 = W_3 \tag{1.2}$$

This will result in an equivalent system

$$\frac{\partial^2 W_1}{\partial t^2} = C_{66}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{2i}{r^2}\frac{\partial}{\partial \varphi} - \frac{1}{r^2}\right) W_1 + C_{55}\frac{\partial^2 W_1}{\partial x_3^2} + \tag{1.3}$$

$$+\frac{C_{11}+C_{12}}{4}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\left[\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_1+\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_2\right]+$$
$$+(C_{13}+C_{55})\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\frac{\partial W_3}{\partial x_3}$$

$$\frac{\partial^2 W_2}{\partial t^2}=C_{66}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}-\frac{2i}{r^2}\frac{\partial}{\partial\varphi}-\frac{1}{r^2}\right)W_2+C_{55}\frac{\partial^2 W_2}{\partial x_3^2}+$$
$$+\frac{C_{11}+C_{12}}{4}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\left[\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_1+\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_2\right]+$$
$$+(C_{13}+C_{55})\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\frac{\partial W_3}{\partial x_3}$$

$$\frac{\partial^2 W_3}{\partial t^2}=C_{55}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)W_3+C_{33}\frac{\partial^2 W_3}{\partial x_3^2}+$$
$$+\frac{C_{13}+C_{55}}{2}\frac{\partial}{\partial x_3}\left[\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_1+\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_2\right]$$

the solution of which will be sought in the form

$$W_1=R_1(r)\,e^{i(k\varphi+\alpha x_3)}\,e^{-pt},\quad W_2=R_2(r)\,e^{i(k\varphi+\alpha x_3)}\,e^{-pt}$$
$$W_3=R_3(r)\,e^{i(k\varphi+\alpha x_3)}\,e^{pt} \tag{1.4}$$

Here k and α are real parameters, while p is a complex ($\operatorname{Re} p \geqslant 0$) one. Substitution of (1.4) into (1.1) yields the following system of ordinary differential equations (1.5)

$$C_{66}\left[R_1''+\frac{1}{r}R_1'-\frac{(k+1)^2}{r^2}R_1\right]-(C_{55}\alpha^2+p^2)R_1+\frac{C_{11}+C_{12}}{4}\left(\frac{d}{dr}-\frac{k}{r}\right)\times$$
$$\times\left[\left(R_1'+\frac{k+1}{r}R_1\right)+\left(R_2'-\frac{k-1}{r}R_2\right)\right]+i\alpha(C_{13}+C_{55})\left(R_3'-\frac{k}{r}R_3\right)=0$$
$$C_{66}\left[R_2''+\frac{1}{r}R_2'-\frac{(k-1)^2}{r^2}R_2\right]-(C_{55}\alpha^2+p^2)R_2+\frac{C_{11}+C_{12}}{4}\left(\frac{d}{dr}+\frac{k}{r}\right)\times$$
$$\times\left[\left(R'_1+\frac{k+1}{r}R_1\right)+\left(R_2'-\frac{k-1}{r}R_2\right)\right]+i\alpha(C_{13}+C_{55})\left(R_3'+\frac{k}{r}R_3\right)=0$$
$$C_{55}\left(R_3''+\frac{1}{r}R_3'-\frac{k^2}{r^2}R_3\right)-(C_{33}\alpha^2+p^2)R_3+\frac{i\alpha(C_{13}+C_{55})}{2}\Big]\left(R_1'+\frac{k+1}{r}R_1\right)+$$
$$+\left(R_2'-\frac{k-1}{r}R_2\right)\Big]=0$$

We shall seek its solution in the form

$$R_1(r)=A_1I_{k+1}(\beta r)+B_1K_{k+1}(\beta r)$$
$$R_2(r)=A_2I_{k-1}(\beta r)+B_2K_{k-1}(\beta r)$$
$$R_3(r)=A_3I_k(\beta r)+B_3K_k(\beta r) \tag{1.6}$$

where $I_\nu(\beta r)$ is a Bessel function with an imaginary argument and $K_\nu(\beta r)$ is a MacDonald function. The resulting system of homogeneous linear algebraic equations for A_i and B_i is

$$[c_{11}\beta^2-(c_{55}\alpha^2+p^2)](A_1+A_2)+2i\alpha\beta(c_{13}+c_{55})A_3=0$$
$$[c_{66}\beta^2-(c_{55}\alpha^2+p^2)](A_1-A_2)=0 \tag{1.7}$$
$$^1/_2\alpha\beta(c_{13}+c_{55})(A_1+A_2)+[c_{55}\beta^2-(c_{33}\alpha^2+p^2)]A_3=0$$

$$[c_{11}\beta^2-(c_{55}\alpha^2+p^2)](B_1+B_2)-2i\alpha\beta(c_{13}+c_{55})B_3=0$$
$$[c_{66}\beta^2-(c_{55}\alpha^2+p^2)](B_1-B_2)=0$$
$$-^1/_2i\alpha\beta(c_{13}+c_{55})(B_1+B_2)+[c_{55}\beta^2-(c_{33}\alpha^2+p^2)]B_3=0 \tag{1.8}$$

Equating its determinant to zero, we obtain the equation for

$$[c_{66}\beta^2 - (c_{55}\alpha^2 + p^2)]\{c_{11}c_{55}\beta^4 - [c_{11}(c_{33}\alpha^2 + p^2) + c_{55}(c_{55}\alpha^2 + p^2) - (c_{13} + c_{55})^2\alpha^2]\beta^2 + (c_{33}\alpha^2 + p^2)(c_{55}\alpha^2 + p^2)\} = 0 \tag{1.9}$$

the pairs of roots of which are given by

$$\beta_1^2 = \frac{C_{55}\alpha^2 + p^2}{C_{66}}$$

$$\beta_{2,3}^2 = \frac{1}{2C_{55}C_{11}}\{C_{55}(C_{55}\alpha^2 + p^2) + C_{11}(C_{33}\alpha^2 + p^2) - \alpha^2(C_{13} + C_{55})^2 + \sqrt{[C_{55}(C_{55}\alpha^2 + p^2) + C_{11}(C_{33}\alpha^2 + p^2) - \alpha^2(C_{13}+C_{55})^2]^2 - 4C_{11}C_{55}(C_{33}\alpha^2+p^2)(C_{55}\alpha^2+p^2)}\} \tag{1.10}$$

which, on substitution into (1.7) and (1.8), yield A_i and B_i.

Particular solution of (1.3) can be written as

$$W_1 = [R_{11}(r) + R_{12}(r)]\, e^{i(k\varphi+\alpha x_3)}e^{pt}, \quad W_2 = [R_{21}(r) + R_{22}(r)]\, e^{i(k\varphi+\alpha x_3)}e^{pt}$$
$$W_3 = [R_{31}(r) + R_{32}(r)]\, e^{i(k\varphi+\alpha x_3)}e^{pt} \tag{1.11}$$

where

$$R_{11}(r) = C_1 I_{k+1}(\beta_1 r) + C_2 I_{k+1}(\beta_2 r) + C_3 I_{k+1}(\beta_3 r)$$
$$R_{21}(r) = -C_1 I_{k-1}(\beta_1 r) + C_2 I_{k-1}(\beta_2 r) + C_3 I_{k-1}(\beta_3 r) \tag{1.12}$$
$$R_{31}(r) = \frac{i[C_{11}\beta_2^2 - (C_{55}\alpha^2 + p^2)]}{\alpha\beta_2(c_{13} + C_{55})} C_2 I_k(\beta_2 r) + \frac{i[C_{11}\beta_3^2 - (C_{55}\alpha^2 + p^2)]}{\alpha\beta_3(C_{13} + C_{55})} C_3 I_k(\beta_3 r)$$

$$R_{12}(r) = B_1 K_{k+1}(\beta_1 r) + B_2 K_{k+1}(\beta_2 r) + B_3 K_{k+1}(\beta_3 r)$$
$$R_{22}(r) = B_1 K_{k-1}(\beta_1 r) + B_2 K_{k-1}(\beta_2 r) + B_3 K_{k+1}(\beta_3 r) \tag{1.13}$$
$$R_{32}(r) = -\frac{i[C_{11}\beta_2^2 - (C_{55}\alpha^2)]}{\alpha\beta_2(C_{13} + C_{55})} B_2 K_2(\beta_2 r) - \frac{i[C_{11}\beta_3^2 - (C_{55}\alpha^2 + p^2]}{\alpha\beta_3} B_3 K_k(\beta_3 r)$$

Solution of the most general form is obtained from (1.11) by summation in k and integration with respect to α and p.

2. As examples, we shall briefly consider solutions of some problems.

We shall assume the initial conditions to be homogeneous, and we shall also assume that the boundary functions admit the Laplace transformation in t, Fourier's transformation in x_3 over the finite or infinite interval and expansion into a Fourier series in terms of the angular coordinate φ.

(*a*) Let us obtain the solution of the second fundamental problem for a hollow cylinder $(a \leqslant r \leqslant b)$ of height h, satisfying the homogeneous initial conditions, and

$$W_j|_{r=a} = f_j(\varphi_1 x_3, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=1}^{\infty} f_j^{(k,m)}(p)\sin\frac{m\pi}{h}x_3\right]dp$$

$$W_j|_{r=b} = \psi_j(\varphi, x_3, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=1}^{\infty} \psi_j^{(k,m)}(p)\sin\frac{m\pi}{h}x_3\right]dp$$

$$W_3|_{r=a} = f_3(\varphi, x_3, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=0}^{\infty} f_3^{(k,m)}(p)\cos\frac{m\pi}{h}x_3\right]dp \tag{2.1}$$

$$W_3|_{r=b}=\psi_3(\varphi, x_3, i)=\frac{2}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=0}^{\infty}\psi_3^{(k,m)}(p)\cos\frac{m\pi}{h}x_3\right]dp$$

To obtain the solution of this problem, we shall utilise the following

$$W_j=\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=1}^{\infty}(R_{j1}(r)+R_{j2}(r)^{(k,m)})\sin\frac{m\pi}{h}x_3\right]dp$$

$$W_3=\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=0}^{\infty}(R_{31}(r)+R_{32}(r)^{(k,m)})\cos\frac{m\pi}{h}x_3\right]dp \tag{2.2}$$

From (2.2) when $r = a$ and $r = b$, together with the boundary conditions (2.1), and taking (1.12) and (1.13) into account, we obtain a set of six algebraic equations

$$[R_{j1}(r)+R_{j2}(r)]_{r=a}^{(k,m)}=f_j^{(k,m)}(p)\,[R_{j1}(r)+R_{j2}(r)]_{r=b}^{(k,m)}=\psi_j^{(k,m)}(p) \qquad (j=1,2,3)$$

which define the arbitrary constants.

Determination of C_jB_j and substitution of obtained values into (2.2) with (1.12) and (1.13) taken into account, completes the general solution of our problem.

(*b*) Let us consider the solution of the first fundamental problem for a solid cylinder ($r \leqslant a$) of height h. Stresses $\sigma_r, \tau_{r\varphi}, \tau_{rx_3}$ will be given in terms of W_i by the following formulas

$$\sigma_r=\frac{c_{11}+c_{12}}{4}\left[\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{i}{r}\right)W_1+\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_2\right]+c_{13}\frac{\partial W_3}{\partial x_3}+$$
$$+\frac{c_{66}}{2}\left[\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}-\frac{1}{r}\right)W_1+\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}-\frac{1}{r}\right)W_2\right] \tag{2.3}$$

$$\tau_{r\varphi}=-\frac{ic_{66}}{2}\left[\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}-\frac{1}{r}\right)W_1-\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}-\frac{1}{r}\right)W_2\right]$$

$$\tau_{rx_3}=\frac{c_{55}}{2}\left(\frac{\partial W_1}{\partial x_3}+\frac{\partial W_2}{\partial x_3}+2\frac{\partial W_3}{\partial r}\right)$$

Let the stresses

$$\sigma_r=f_1(\varphi, x_3, t)=\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=1}^{\infty} f_1^{(k,m)}(p)\sin\frac{m\pi}{h}x_3\right]dp$$

$$\tau_{r\varphi}=f_2(\varphi, x_3, t)-\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} c^{ik\varphi}\sum_{m=1}^{m} f_2^{(k,m)}(p)\sin\frac{m\pi}{h}x_3\right]dp \tag{2.4}$$

$$\tau_{rx_3}=f_3(\varphi, x, t)=\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\left[\sum_{k=-\infty}^{\infty} e^{ik\varphi}\sum_{m=0}^{\infty} f_3^{(k,m)}(p)\cos\frac{m\pi}{h}x_3\right]dp$$

be given on the surface of the cylinder. We shall use the formulas (2.2) and (2.3) with $R_{j2}(r) = 0$ ($j = 1, 2, 3$) to obtain the solution of our problem, and

$$\left\{\frac{c_{11}+c_{12}}{4}\left[\left(R_{11}'+\frac{k+1}{r}R_{11}\right)+\left(R_{21}'-\frac{k-1}{r}R_{21}\right)\right]+i\alpha c_{13}R_{31}+\frac{c_{66}}{2}\left[\left(R_{11}'-\frac{k+1}{r}R_{11}\right)+\right.\right.$$
$$\left.\left.+\left(R_{21}'+\frac{k-1}{r}R_{21}\right)\right]\right\}_{r=a}=f_1^{(k,m)}(\alpha) \tag{2.5}$$

$$-\frac{ic_{66}}{2}\left[\left(R_{11}' - \frac{k+1}{r} R_{11}\right) - \left(R_{21}' + \frac{k-1}{r} R_{21}\right)\right]_{r=a} = f_2^{(k,\,m)}(\alpha),$$

$$\frac{c_{55}}{2}\left[i\alpha\,(R_{11} + R_{21}) + 2R_{31}'\right]_{r=a} = f^{(k,\,m)}(\alpha)$$

to find three arbitrary constants c_i. Insertion of obtained values into (2.2) completes the general solution of our problem. The same method can be used to solve other dynamic problems for a transversely isotropic, elastic cylinder.

In case of steady oscillations, integrals of the type

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,dp \qquad (2.6)$$

should, wherever they occur, be replaced with the factor $e^{i\omega t}$. With $p = 0$, (2.1) yields the solutions of the corresponding static problems.

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